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B. Lukács

ALL VACUUM METRICS WITH SPACE-LIKE SYMMETRY AND SHEARING GEODESIC TIMELIKE EIGENRAYS

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

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ALL VACUUM METRICS WITH SPACE-LIKE SYMMETRY AND SHEARING GEODESIC TIMELIKE EIGENRAYS

B. Lukács
High Energy Physics Department
Central Research Institute for Physics, Budapest, Hungary

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ABSTRACT

This paper contains the general solution of the field equations of space-like symmetric vacuum gravitational fields having shearing geodesic timelike eigenrays. The rotation of the eigenrays vanishes except the trivial Minkowskian solution. The solutions divide into two classes. One of them contains solutions with functional dependence among the field quantities /Papapetrou solutions and the Kasner solution./ The other class contains new solutions. The behaviour of these solutions is ill, probably they have not physical importance.

These solutions are the space-like symmetric analogues of the Kóta-Perjés solutions.

KIVONAT

Ez a munka az Einstein-egyenletek összes olyan, térszerüen szimmetrikus vákuummegoldását tartalmazza, melynek sajátsugarai időszerüek, nyiróak és geodetikusak. A sajátsugarak rotációja minden ilyen megoldásra eltünik, kivéve a Minkowski-teret. A megoldásoknak két csoportja van. Az egyikbe azok tartoznak, amelyeknél a térmennyiségek között funkcionális függés van /Papapetrou-féle megoldások, valamint a Kasner-megoldás/. A másik csoportba uj megoldások tartoznak. A megoldások aszimptotikus viselkedése rossz.

Ezek a megoldások a Kóta-Perjés megoldások megfelelői térszerü szimmetria esetén.

РЕЗЮМЕ

Работа содержит все возможные вакуумные решения уравнений Эйнштейна для пространственных симметричных гравитационных полей, имеющих временные срезающие геодетические собственные лучи. При таких решениях собственные лучи обладают ротацией только в случае поля Минковского. Решения разделяются на две группы. Одна из них содержит решения, при которых между пространственными величинами существует функциональная зависимость /решения Папапетру и решение Каснера/. Во вторую группу входят новые решения. Асимптотическое поведение этих решений плохое.

Рассматриваемые решения, полученные для пространственных симметричных полей, аналогичны решениям Кота-Перйеш.

1. INTRODUCTION

The general solutions of the Einstein-equations is unknown. Different methods have been applied to find solutions of these equations. The spin coefficient technique developed by Newman and Penrose [1] is one of them. The Newman-Penrose equations can be solved exactly when the gravitational field possesses geodesic rays [2]. The solutions having shearing geodesic rays are dubious for physical interpretation. The class of solutions with nonshearing geodesic rays contains several interesting solutions, for example the Kerr solution and the plane wave solution.

When the space-time has a /time-like or space-like/ symmetry, the gravitational equations can be reformulated in a 3 dimensional /background/ space [3]. We obtain Einstein-equations in this 3 dimensional space with energy-momentum tensor of a complex "gravitational" 3-vector field $\underline{G}$ and field equations for $\underline{G}$. The spin coefficient technique can be applied in this 3 dimensional space but the analogues of Newman-Penrose-equations have simpler form than the 4 dimensional equations.

In the 3 dimensional formalism it is suitable to introduce the notion of eigenrays [3], [4] as an analogues of the rays. When the rays have no shearing, the eigenrays are also nonshearing and they coincide with the 3 dimensional projections of rays. /When the rays have shearing, the connection is complicated./ Furthermore when the rays are geodesic and nonshearing, the eigenrays are also geodesic and nonshearing in the background space [3]. Thus the solutions having geodesic nonshearing eigenrays are known as the exact solutions of the 4 dimensional Newman-Penrose equations with a symmetry. On the other hand in stationary vacuum case it is known that the equations of 3 dimensional relativity can be solved exactly if the eigenrays are

a/ geodesic and shearing, [5]

b/ nongeodesic and nonshearing [6].

In these cases the rays are nongeodesic. /Nevertheless these solutions are dubious for physical interpretation./ Now we will show that the equations of the space-like symmetric problem also can be solved when the eigenrays are timelike, shearing and geodesic.

Let the coordinate $x^3 = z$ be chosen as the arc of the trajectories of the Killing motion. The line element can be written as follows:

$$d\tilde{s}^2 = -f^{-1}\, ds^2 + f\left(dz + \omega_i\, dx^i\right)^2$$

$$i = 0, 1, 2 \qquad /1.1/$$

$$ds^2 \equiv g_{ik}\, dx^i\, dx^k .$$

The quantities f, ω_i, g_{ik} are independent of z. g_{ik} is the metric tensor of the background space. Making use of the line element /1.1/ when writing down the 4 dimensional vacuum Einstein-equations we obtain the following equations: [3]

$$G^r_{\ |r} + \left(\bar{G}^r - G^r\right) G_r = 0 \ ;$$

$$G_{i|k} - G_{k|i} + G_i \bar{G}_k - \bar{G}_i G_k = 0 \ ; \qquad /1.2/$$

$$R_{ik} + G_i \bar{G}_k + \bar{G}_i G_k = 0 .$$

Here

$$G_i \equiv \frac{f_{,i} + i\varphi_i}{2f} \ ; \quad \varphi_i \equiv e_{ik\ell}\, \omega^{k|\ell} \sqrt{g}\, f^2 \qquad /1.3/$$

and R_{ik} is the Ricci tensor of the background space.

One can introduce in the background space a complex basic vector triad $\underline{z}_p = /\underline{\ell}, \underline{m}, \underline{\bar{m}}/$, $p = 0, +, -$ with the orthonormalization relations: $\underline{\ell}\underline{\ell} = -\underline{m}\underline{\bar{m}} = \ell$; $\underline{\ell}\underline{m} = \underline{m}\underline{m} = 0$. If we fix the direction of vector $\underline{\ell}$ by the relation

$$G_- \equiv \bar{m}^i\, G_i = 0 \qquad /1.4/$$

then $\underline{\ell}$ is the tangent vector of the eigenrays of the gravitational field and the eigenrays are timelike [4]. /The vector $\underline{\ell}$ can be always chosen in this way except three special cases [4]. In these three cases the solutions have lightlike or spacelike eigenrays. Here we do not deal with these cases./ We can take the coordinate system in the following way:

$$\ell^i = \delta^i_o \ ; \quad m^i = \omega\delta^i_o + \xi^{\underline{a}}\, \delta^i_{\underline{a}} \ ; \quad \underline{a} = 1,2 \qquad /1.5/$$

and the transformational freedom

$$t' = t + t^o(x^{\underline{a}})$$
$$x^{\underline{a}'} = x^{\underline{a}'}(x^{\underline{b}}) \qquad /1.6/$$

remains. We define the rotational coefficients similarly to Ref. 3. Then we can make the quantity ϵ zero and still there is the following freedom in the choice of triad: [4]

$$\underline{\ell}' = \underline{\ell} \ ;$$
$$\underline{m}' = e^{iC^o}\, \underline{m} \qquad /1.7/$$
$$C^o \text{ is real and } DC^o = 0 \ .$$

Now we write down the triad components of the eqs. /1.2/ for the case of geodesic eigenrays using the triad choice $\epsilon = 0$ [4].

$$D\rho = -\rho^2 - \sigma\bar{\sigma} - G_o\bar{G}_o$$

$$D\sigma = -(\rho+\bar{\rho})\sigma$$

$$D\tau = -\rho\tau + \bar{\sigma}\bar{\tau} - G_o\bar{G}_-$$

$$\delta\rho - \bar{\delta}\sigma = 2\sigma\tau - \bar{G}_o G_+$$

$$\delta\tau + \bar{\delta}\bar{\tau} = -2\tau\bar{\tau} - \sigma\bar{\sigma} + \rho\bar{\rho} - G_o\bar{G}_o - G_+\bar{G}_-$$

$$DG_o = \left(-2\bar{\rho} + G_o - \bar{G}_o\right)G_o \qquad /1.8/$$

$$\delta G_o - DG_+ = \left(\bar{\rho}+\bar{G}_o\right)G_+$$

$$\bar{\delta}G_o = \bar{\sigma}G_+ - \bar{G}_- G_o$$

$$\bar{\delta}G_+ = -\left(\tau+\bar{G}_-\right)G_+ + (\rho-\bar{\rho})G_o$$

$$D \equiv \frac{\partial}{\partial t} \ ; \qquad \delta \equiv \omega\frac{\partial}{\partial t} + \xi^{\underline{a}}\frac{\partial}{\partial x^{\underline{a}}}$$

The commutators of $D, \delta, \bar{\delta}$ are:

$$D\delta - \delta D = -\bar{\rho}\delta - \sigma\bar{\delta}$$
$$\delta\bar{\delta} - \bar{\delta}\delta = \tau\delta - \bar{\tau}\bar{\delta} - (\rho - \bar{\rho})D \qquad /1.9/$$

In the following Sections we solve these equations. We remark that in several points the procedure of integration is similar to the procedure written down in Ref. 5. In these points we omit the details and refer to Ref. 5.

2. THE ANALOGUES OF THE KÓTA-PERJÉS THEOREM

We prove the following analogue of the Kóta-Perjés theorem:

Timelike geodesic eigenrays of a curved space-like symmetric vacuum space-time cannot have coexisting shearing and curl. If the eigenrays have shearing, then they also have divergence and the following relation is valid:

$$\rho\bar{\rho} - \sigma\bar{\sigma} - G_o\bar{G}_o = 0 \quad . \qquad /2.1/$$

First, we observe that

$$D(\sigma/\bar{\sigma}) = 0 \qquad /2.2/$$

according to eq. /1.8b/. Using the freedom /1.7/ we can make σ real and positive.

If $G_o = 0$, it is seen from eq. /1.8h/ that $G_+ = 0$. But $\underline{G} = 0$ means that the space-time is Minkowskian according to eqs. /1.2/. We abandon this case.

Applying the commutator /1.9a/ on the quantity $\ln G_o$ we get:

$$\delta(\ln G_o\sigma) = G_+ - 2\tau \; . \qquad /2.3/$$

According to eqs. /1.8f/, /2.3/ the propagation properties of

$$\gamma^2 \equiv G_o\bar{G}_o \quad \text{are}$$

$$D\gamma^2 = -2(\rho+\bar{\rho})\gamma^2 \qquad /2.4/$$
$$\delta\gamma^2 = \sigma G_o\bar{G}_- - \gamma^2(\delta\ln\sigma + 2\bar{\tau}) \quad .$$

Now applying the operator D on eq. /2.3/ we get the same equation as in Ref. 5.

$$\gamma\left(3\bar{\delta}\rho + \bar{\delta}\bar{\rho} + 2\delta\sigma\right) + 2\sigma\delta\gamma = 0 \quad . \qquad /2.5/$$

We may introduce the following new operators:

$$\delta_{\pm} \equiv R\left(\delta \pm i\bar{\delta}\right)$$
$$DR \equiv \frac{\rho+\bar{\rho}}{2} R \qquad /2.6/$$

and, from eq. /2.5/, we obtain:

$$\delta(\rho+\bar{\rho}) = \delta\left(\sigma^2 + \gamma^2 + \frac{1}{4}(\rho-\bar{\rho})^2\right) = 0 \qquad /2.7/$$

in the same way as in Ref. 5.

Applying the commutator /1.9b/ on the quantity $\rho+\bar{\rho}$ we get:

$$(\rho+\bar{\rho})(\rho-\bar{\rho}) = 0 \quad . \qquad /2.8/$$

Let us write G_o in the following form:

$$G_o = \gamma e^{i\chi} \quad . \qquad /2.9/$$

Applying the commutator /1.9b/ on G_o and using the eqs. /2.3/, /2.4/, /2.9/ we obtain:

$$(\rho-\bar{\rho})^2 + 2\left(\rho\bar{\rho} - \sigma^2 - \gamma^2\right) = 0 \quad . \qquad /2.10/$$

From the eqs. /2.8/, /2.10/ we see that

$$\rho = \bar{\rho} \neq 0 \quad . \qquad /2.11/$$

The eqs. /2.5/, /2.7/, /2.10/, /2.11/ give the following equations:

$$\text{If } \sigma \neq 0: \quad \begin{aligned} &\mathrm{Re}\rho \neq 0 \ ; \\ &\mathrm{Im}\rho = 0 \ ; \\ &\rho\bar{\rho} - \sigma\bar{\sigma} - G_o\bar{G}_o = 0 \ ; \\ &\delta\rho = \delta\bar{\rho} = \delta\sigma = \delta\gamma = 0 \ . \end{aligned} \qquad /2.12/$$

Thus we completely proved the Theorem.

We can get equations for the quantities ω, $\xi^{\underline{a}}$ as well applying the commutators /1.9/ on the quantities t, $x^{\underline{a}}$ and using eqs. /2.12/:

$$A \equiv \begin{pmatrix} \omega \\ \xi^{\underline{a}} \end{pmatrix} ; \quad \begin{aligned} &DA = -\rho A - \sigma \bar{A} \\ &\delta \bar{A} - \bar{\delta} A = \tau A - \bar{\tau} \bar{A} \ . \end{aligned} \qquad /2.13/$$

It is easy to integrate the eqs. /1.8a-b,f/, /2.13/ take eqs. /2.12/ into account. The results are the following:

$$\rho = \frac{\sigma}{\sigma^o} = \frac{\gamma}{\gamma^o} = \frac{1}{2t} \ ; \qquad \sigma^{o2} + \gamma^{o2} = 1 \ ;$$

$$G_o = -\frac{\gamma^o}{2t} \frac{t^{\gamma^o} - iQ}{t^{\gamma^o} + iQ} \ ; \qquad \sigma^o, \gamma^o \text{ are constant}$$

$$\xi^{\underline{a}} = \frac{1}{\sqrt{2t}} \left[A^{\underline{a}} t^{-\sigma^o/2} + iB^{\underline{a}} t^{\sigma^o/2} \right] \ ; \qquad Q_{,t} = 0 \ ; \qquad /2.14/$$

$$\omega = 0 \ .$$

$$\varepsilon \equiv f + i\varphi = i\varphi^o + \frac{f^o}{t^{\gamma^o} + iQ} \ .$$

The further equations give:

$$\bar{\delta}\varepsilon = 0 \ ;$$

$$2\sigma\tau = \bar{G}_o G_+ \ ;$$

$$\tau(\sigma^2 - \gamma^2) = 0 \ ; \qquad /2.15/$$

$$\mathrm{Im}(\delta + \bar{\tau})\bar{\xi}^{\underline{a}} = 0 \ .$$

The eq. /2.15c/ has two solutions: $\tau = 0$ or $\sigma^o = \gamma^o = 1/\sqrt{2}$. We must deal with these two cases separately.

3. SOLUTIONS WITH $\tau = 0$

In this case the eqs. /2.15/ can be easily solved. /2.15a,b/ show that $\varphi^o = 0$, f^o and Q are constant. According to /2.15d/ we can choose the coordinates $x^2 = x$, $x^3 = y$ in such a way that

$$A^{\underline{a}} = \delta^{\underline{a}}_2 \ ; \qquad B^{\underline{a}} = \delta^{\underline{a}}_3 \ . \qquad /3.1/$$

Thus

$$ds^2 = dt^2 - t^{1+\sigma^o} dx^2 - t^{1-\sigma^o} dy^2 \quad . \qquad /3.2/$$

It is easy to calculate f and ω_i in the same way as in Ref. 5.

4. METRICS WITH $\tau \neq 0$, $\sigma^o = \gamma^o$

The method of the integration of eqs. /2.15/ is analogous to the procedure written down in Ref. 5. We introduce new differential operators by the following definitions:

$$\hat{\alpha} \equiv \sqrt{-f^o}\, B^{\underline{a}} \frac{\partial}{\partial x^{\underline{a}}} \ ; \quad \hat{\beta} \equiv \sqrt{-f^o}\left(A^{\underline{a}} - QB^{\underline{a}}\right) \frac{\partial}{\partial x^{\underline{a}}} \qquad /4.1/$$

From the eqs. /2.15/ we get:

$$\begin{aligned} [\hat{\alpha},\hat{\beta}] &= -2(\hat{\alpha}Q)\alpha \\ \hat{\alpha}Q &= \hat{\beta}\ln(-f^o) \\ \hat{\beta}Q &= 0 \\ \hat{\alpha}\varphi^o &= 0 \\ \hat{\beta}\varphi^o &= \hat{\alpha}f^o \end{aligned} \qquad /4.2/$$

Applying the commutator $[\hat{\alpha},\hat{\beta}]$ on Q we find that

$$\hat{\beta}\left(f^{o^{-3}}\ \hat{\beta}\ f^o\right) = 0 \quad . \qquad /4.3/$$

Comparing this equation with /4.2c/ we see that there are two different cases: either Q = constant or $\left(f^{o^{-3}}\ \hat{\beta}\ f^o\right)$ is a functional of Q. These two cases can be treated similarly to Ref. 5.

5. THE FINAL RESULTS

The reconstruction of the 4 dimensional line element is a simple procedure and it happens similarly to the one written down in Ref. 5. The 3 different line elements can be seen on Table 1.

6. PHYSICAL INTERPRETATION

These metrics are spherics [4], but their behaviour is awkward. Investigating the curvature quantities ψ_a [4] we get the following results:

Case a/: $\tau = 0$

The curvature quantities are independent of x, y. This spacetime has true singularity at $t = 0$. In the limit $t \to \infty$ f vanishes. Since both f and φ depend on t only, this metric is a Papapetrou-type solution, except case $Q = 0$, when this metric is a Kasner solution. [7]

Case b/: $\tau \neq 0$, $Q =$ constant

This space-time contains true singularity at $t = 0$. If t is fixed and x or y goes to infinity, the curvature quantities do not vanish. In the limit $t \to \infty$ while x,y is fixed, f vanishes.

Case c/: $\tau \neq 0$, $Q \neq$ constant.

This space-time contains true singularities in the following hypersurfaces: $t = 0$, x and y are arbitrary; $ay + b = 0$, t and x are arbitrary. In the limit $t \to \infty$ while x and y are fixed, f vanishes. In the limit when t is fixed and x goes to infinity, the curvature quantities become infinite. If t is fixed and y goes to infinity, f vanishes.

The behaviour of these solutions indicates that these solutions have no physical importance.

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TABLE 1.

METRICS WITH $\sigma \neq 0$, $\kappa = 0$.

$\underline{\tau = 0}$

$$d\tilde{s}^2 = \frac{f^o t^{\gamma^o}}{t^{2\gamma^o} + Q^2}\left(dz - \frac{2\gamma^o Q}{f^o} x\, dy\right)^2 - \frac{t^{2\gamma^o} + Q^2}{f^o\, t^{\gamma^o}}\left(dt^2 - t^{1+\sigma^o} dx^2 - t^{1-\sigma^o} dy^2\right)$$

f^o, σ^o, Q are constant, $f^o < 0$, $\gamma^o = \sqrt{1-\sigma^{o2}}$.

$\underline{\tau \neq 0,\ Q = \text{constant}}$

$$d\tilde{s}^2 = \frac{P(x+Qy)t^{\sigma^o}}{t^{2\sigma^o} + Q^2}\left(dz + 2\sigma^o Qy\, dx\right)^2 - 2dt\left(dz + 2\sigma^o Qy\, dx\right) -$$

$$- \left(t^{2\sigma^o} + Q^2\right)\left(t^{1-2\sigma^o} dx^2 + t\, dy^2\right)$$

P is a negative constant, $\sigma^o = 1/\sqrt{2}$.

$\underline{\tau \neq 0,\ Q \neq \text{constant}}$

$$d\tilde{s}^2 = -\frac{t^{\sigma^o} x}{t^{2\sigma^o} + y^2}\left[dz + \frac{\sigma^o y}{2}\frac{(ay+b)^4}{k^2 x^4} dx\right]^2 - dt\left[2dz + \sigma^o y\frac{(ay+b)^4}{k^2 x^4} dx\right] -$$

$$- \frac{\left(t^{2\sigma^o} + y^2\right)(ay+b)^2}{k^2 x^6 t^{2\sigma^o} - 1}\Big\{(ay+b)^2 \left(t^{2\sigma^o} + y^2\right)dx^2 +$$

$$+ 2x(ay+b)\left[at^{2\sigma^o} + (2ay+b)y\right] dx\, dy +$$

$$+ x^2\left[a^2 t^{2\sigma^o} + (2ay+b)^2\right] d^2y\Big\}$$

k, a, b are real contants.

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